

On cubic Berwald spaces

Nicoleta Brinzei
Transilvania University, Brasov, Romania

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Abstract

We show that, for Finsler spaces with cubic metric, Landsberg spaces are Berwaldian. Also, for decomposable metrics, we determine specific conditions for a space with cubic metric to be of Berwald type, thus refining the result in [6].

1 Introduction

Spaces with cubic metric are studied by Matsumoto and Numata, [6], [7]. They are Finsler spaces in a wider sense, [9].

An interesting problem related to m -th root metric spaces is the following: is any Landsberg space with m -th root metric Berwaldian?

A partial answer for spaces with cubic metric with fundamental function $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ (where α^2 is a pseudo-Riemannian metric and β is a 1-form) is given by Lee and Jun, [5]. In what follows, we generalize this result: namely, for all cubic Finsler spaces (M, F) , $F = \sqrt[3]{a_{ijk}(x)y^iy^jy^k}$ with a_{ijk} differentiable, if (M, F) is of Landsberg type, then it is of Berwald type.

Also, for spaces whose fundamental function is decomposable as a product of two factors $\bar{F}^3 = a \cdot b$, between a Riemannian metric a and a 1-form b on M , we show that (M, \bar{F}) is of Berwald type if and only if the 1-form b is parallelly transported with respect to the Levi-Civita connection of a . An analogous result is proven by Z. Shen for spaces with (α, β) -metrics of the form $F = \alpha\phi(\frac{\beta}{\alpha})$, [11].

The techniques we used mainly rely on expressing the involved geometrical objects in terms of the third power $T = F^3$ of the fundamental function, which is a polynomial function of the directional variables y^i .

2 Spaces with cubic metric

Let M^n be a differentiable manifold of dimension n and class C^∞ , TM its tangent bundle and (x^i, y^i) the coordinates in a local chart on TM . Let F be

the following function on M , :

$$F = \sqrt[3]{a_{ijk}(x)y^i y^j y^k}. \quad (1)$$

(with a_{ijk} symmetric in all its indices) and

$$T = F^3 = a_{ijk}(x)y^i y^j y^k. \quad (2)$$

In the following, for a function $f = f(x, y)$, we shall denote by " , " and " . " the partial derivatives w.r.t. x and y , respectively. Also, if N is a nonlinear connection on TM , we denote by " ; " its associate covariant derivative

$$f_{;l} = \frac{\delta f}{\delta x^l} = \frac{\partial f}{\partial x^l} - N_l^r \frac{\partial f}{\partial y^r}, \quad f \in \mathcal{F}(TM)$$

and we denote by null index transvection by y (for instance, $T_{i0} = T_{ij}y^j$).

Remark 1 [3] If $F = T^{1/m}$ is a Finslerian fundamental function on M , then the Hessian $[T_{ij}]$ is an invertible matrix, its inverse has the entries:

$$T^{ij} = \frac{1}{m(m-1)F^{m-2}} \{(m-1)g^{ij} - (m-2)l^i l^j\},$$

where g^{ij} denotes the contravariant version of the usual Finslerian metric tensor attached to F and $l^i = \frac{y^i}{F}$.

Hence, T^{ij} and T_{ij} can be used for raising and lowering indices of tensors. Moreover, T_{ij} are polynomial functions of y , and T^{ij} are rational functions of y .

3 Geodesics and canonical spray

In the following, we shall express the equations of geodesics of a cubic metric space and the related geometric objects in terms of $T = F^3$ of the fundamental function and of its derivatives.

Unit speed geodesics of (M, F) are described by the Euler-Lagrange equation:

$$\frac{\partial F}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial F}{\partial y^i} \right) = 0.$$

Taking into account the fact that, along such curves, $F(x, \dot{x}) = 1$, the above is equivalent to:

$$\frac{\partial T}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial T}{\partial y^i} \right) = 0.$$

An easy computation leads to:

$$\frac{dy^i}{dt} + T^{ih}(T_{h,k}y^k - T_{;h}) = 0, \quad y^i = \dot{x}^i. \quad (3)$$

Consequently,

Proposition 2 1. In spaces with cubic metric the coefficients of the canonical spray, [1], [8], are rational functions of (y^i) , given by

$$2G^i = T^{ih}(T_{h,k}y^k - T_{,h}). \quad (4)$$

2. The canonical nonlinear connection has the coefficients: $N^i_{\cdot j} = G^i_{\cdot j} = \frac{1}{2}\{T^{ih}_{\cdot j}(T_{h,k}y^k - T_{,h}) + T^{ih}(T_{hj,k}y^k + T_{h,j} - T_{j,h})\}$.

We denote in the following by $B\Gamma$ the Berwald connection, [1], [2] determined by $F = \sqrt[3]{T}$ and by $G^i_{jk} = G^i_{\cdot jk}$ its coefficients. According to (4), for m-th root metric spaces, G^i_{jk} are rational functions of y .

Also, let

$$\begin{aligned} L^i_{jk} &= \frac{T^{ih}}{2} \left(\frac{\delta T_{hj}}{\delta x^k} + \frac{\delta T_{hk}}{\delta x^j} - \frac{\delta T_{jk}}{\delta x^h} \right), \\ T^i_{jk} &= \frac{T^{ih}}{2} \left(\frac{\partial T_{hj}}{\partial y^k} + \frac{\partial T_{hk}}{\partial y^j} - \frac{\partial T_{jk}}{\partial y^h} \right) = \frac{T^{ih}}{2} T_{hjk}. \end{aligned}$$

denote the coefficients of the canonical metrical connection CT attached to the Lagrange-type metric T_{ij} , [8].

4 Specific Landsberg&Berwald conditions for mth- root metrics

There are a lot of alternative definitions of Landsberg and Berwald-type Finsler spaces, [1], [4]. In the present paper, we shall use the following:

A Finsler space (M, F) is a *Landsberg space* if: (1) the Cartan tensor C_{ijk} satisfies $C_{ijk|0} = 0$, where the covariant derivative is taken with respect to the Berwald connection $B\Gamma$, or (2): the Berwald connection $B\Gamma$ is metrical.

In Landsberg spaces, the horizontal coefficients of the Cartan connection F^i_{jk} coincide with those of the Berwald connection: $F^i_{jk} = G^i_{jk}$.

A Finsler space is called a *Berwald space* if: (1) with respect to $B\Gamma(N)$, there holds $C_{ijk|l} = 0$ or (2) the coefficients G^i_{jk} of the Berwald connection are functions of x^i alone: $G^i_{jk} = G^i_{jk}(x)$.

The last statement is equivalent to the fact that the coefficients G^i of the canonical spray are homogeneous polynomial functions of degree 2 in y^i . There hold the inclusions:

$$Riemann\ spaces \subset Berwald\ spaces \subset Landsberg\ spaces.$$

For Finsler spaces with m-th root metric (M, F) , we get more convenient such characterizations by using the third order derivatives T_{ijk} (where $T = F^m$) instead of the Cartan tensor C_{ijk} .

Using the results in [10], we have proven in [3], that

Proposition 3 *The horizontal coefficients L^i_{jk} of the canonical metrical connection CT attached to the Hessian T_{ij} coincide with those of the Cartan connection of (M, F) . Hence, in Landsberg m -th root metric spaces, we have $L^i_{jk} = F^i_{jk} = G^i_{jk}$.*

Corollary 4 *An m -th root metric space (M, F) is a Berwald space (resp. Landsberg space) if and only if, w.r.t. the canonical metrical connection $CT(N)$, we have $T_{ijk|l} = 0$ (resp. $T_{ijk|0} = 0$).*

5 Landsberg-Berwald equivalence

In the following, we show that Landsberg spaces with cubic metrics are Berwaldian.

Let

$$T = F^3 = a_{ijk}(x)y^i y^j y^k,$$

with $a_{ijk} = a_{ijk}(x)$ of class at least 1, define a Landsberg space; according to the results in the previous section, this means

$$T_{ijk|0} = 0.$$

For a cubic metric, the third derivatives T_{ijk} depend only on x , which entails $\frac{\delta T_{ijk}}{\delta x^l} = \frac{\partial T_{ijk}}{\partial x^l}$.
Then,

$$T_{ijk|l} = T_{ijk,l} - L^h_{il} T_{hjk} - L^h_{jl} T_{ihk} - L^h_{kl} T_{ijh}. \quad (5)$$

Taking into account that our space is a Landsberg one (i.e., $L^h_{il} = G^h_{il}$ etc.), we have

$$T_{ijk|0} = T_{ijk,l} y^l - N^h_i T_{hjk} - N^h_j T_{ihk} - N^h_k T_{ijh} = 0.$$

Deriving by y^l and taking into account that T_{ijk} depend only on x , we get

$$T_{ijk,l} - L^h_{il} T_{hjk} - L^h_{jl} T_{ihk} - L^h_{kl} T_{ijh} = 0,$$

which is nothing but $T_{ijk|l} = 0$. We have thus obtained

Proposition 5 *Let (M, F) be a space with cubic metric $F = \sqrt[3]{a_{ijk}(x)y^i y^j y^k}$. If the functions a_{ijk} are of class at least one, then there holds the implication:*

$$(M, F) \text{ is a Landsberg space} \Rightarrow (M, F) \text{ is a Berwald space.}$$

Further, for spaces with cubic metric, the inclusion Riemannian spaces \subset Berwald spaces is strict. Namely, the Berwald-Moor conformal space with

$$T = F^3 = e^{\sigma(x)} y^1 y^2 y^3,$$

where $\sigma(x)$ is a differentiable function, provides an example of Berwald cubic space, which is non-Riemannian.

6 Decomposable cubic metrics

Let us consider a space $(M, F = \sqrt[3]{T})$, where T decomposes as a product

$$T = a \cdot b \quad (6)$$

where $a = \gamma_{ij}(x)y^i y^j$ is a Riemannian metric and $b = b_i(x)$ is a 1-form, such that:

$$\|b\|^2 = \gamma^{ij} b_i b_j = 1.$$

For cubic spaces with $T = F^3$ as in (6), it is proven in [6] that the space is a Berwald one if and only if there exists some 1-form $f \in \mathcal{X}^* M$ such that

$$\gamma_{ij|k} = f_k(x) \gamma_{ij}; \quad b_{i|k} = -f_k(x) b_i,$$

where the covariant derivative is taken with respect to the Berwald connection determined by the "whole" fundamental function $F = \sqrt[3]{ab}$.

In the following, we shall find the relation between a and b such that the space $(M, F = \sqrt[3]{ab})$ is Berwaldian; more precisely, we shall take into consideration the covariant derivatives

$$\nabla_i b_j,$$

where ∇ denotes the Levi-Civita connection attached to γ_{ij} .

By direct computation, we get

Lemma 6 *If $a = \gamma_{ij}(x)y^i y^j$ is a Riemannian metric and $b = b_i(x)$ is a 1-form with $\gamma^{ij} b_i b_j = 1$, then:*

1. *The Hessian matrix $[T_{ij}]$ is invertible iff*

$$\Delta := 4b^2 - a$$

does not vanish.

2. *The inverse matrix has the entries*

$$T^{ij} = \frac{1}{2b\Delta} (\Delta \gamma^{ij} - 2bb^i y^j - 2bb^j y^i + ab^i b^j + y^i y^j), \quad (7)$$

where the indices of b were raised by $\gamma^{ih} : b^i = \gamma^{ih} b_h$.

Furhter, in [1], p. 110-111, it is proven the following result:

Lemma 7 , [1]: *If (M, F) and (M, \bar{F}) are two Finsler spaces on the same underlying manifold, then the local coefficients of the corresponding canonical sprays are related by*

$$2\bar{G}^i = 2G^i + \frac{\bar{F}_{|0} y^i}{F} - \bar{F} \bar{g}^{ij} r_j(\bar{F}), \quad (8)$$

where $|$ denotes Berwald covariant derivative determined by F and

$$r_j(S) = S_{|j} - y^r S_{|r \cdot j}, \quad \forall S \in \mathcal{F}(TM).$$

In the following, we shall express the above in terms of the m -th power of \bar{F} , $m \geq 2$; hence, let for the moment

$$T = \bar{F}^m.$$

Then, there hold the relations:

•

$$\frac{\bar{F}_{|0} y^i}{\bar{F}} = \frac{1}{m} \frac{T_{|0} y^i}{T}. \quad (9)$$

• The contravariant Finslerian metric tensor \bar{g}^{ij} is expressed in terms of T as

$$\bar{g}^{ij} = \frac{T^{-\frac{2}{m}}}{m-1} (Tm(m-1)T^{ij} + (m-2)y^i y^j).$$

• $r_j(\bar{F}) = \frac{1}{m} T^{\frac{1}{m}-2} \left(T r_j(T) + \frac{m-1}{m} T_j T_{|0} \right);$

• $y^j r_j(T) = (1-m)T_{|0}.$

Then, the last term in (8) is

$$\begin{aligned} \bar{F} \bar{g}^{ij} r_j(\bar{F}) &= T^{\frac{1}{m}} \frac{T^{-\frac{2}{m}}}{m-1} (Tm(m-1)T^{ij} + (m-2)y^i y^j) \cdot \\ &\cdot \frac{1}{m} T^{\frac{1}{m}-2} \left(T r_j(T) + \frac{m-1}{m} T_j T_{|0} \right) = \\ &= \frac{T^{-2}}{m(m-1)} \{ T^2 m(m-1) T^{ij} r_j(T) + (m-2) y^i y^j T r_j(T) + (m-1)^2 T T^{ij} T_j T_{|0} + \\ &+ \frac{(m-2)(m-1)}{m} y^i y^j T_j T_{|0} \} = T^{ij} r_j(T) - \frac{m-2}{m} T^{-1} y^i T_{|0} + \frac{1}{m} T^{-1} T_{|0} y^i + \\ &+ \frac{m-2}{m} T^{-1} y^i T_{|0} = T^{ij} r_j(T) + \frac{1}{m} T^{-1} T_{|0} y^i. \end{aligned}$$

Replacing into (8) and taking (9) into account, we get

Lemma 8 *If (M, F) and (M, \bar{F}) are two Finsler spaces on the same underlying manifold, then the coefficients of the corresponding canonical sprays are related by*

$$2\bar{G}^i = 2G^i - T^{ij} r_j(T), \quad (10)$$

where $|$ denotes Berwald covariant derivative determined by F and

$$T = \bar{F}^m, \quad m \geq 2, \quad r_j(T) = T_{|j} - y^r T_{|r \cdot j}.$$

We shall also use the following relations, which can be deduced by direct computation:

$$\begin{aligned}
r_j(b) &= (\nabla_j b_r - \nabla_r b_j) y^r; \\
y^j r_j(b) &= 0; \\
T^{ij} b_j &= \frac{1}{2\Delta} (2bb^i - y^i); \quad T^{ij} a_{.j} = \frac{1}{\Delta} (2by^i - b^i a); \\
\|b\| &= 1 \Rightarrow b^i \nabla_j b_i = 0.
\end{aligned} \tag{11}$$

Let now G^i be determined by the Riemannian metric $\gamma_{ij}(x)$, where $a = \gamma_{ij}(x) y^i y^j$, and \bar{G}^i , by $T = \bar{F}^3 = a \cdot b$ as above. Then, $|_i = \nabla_i$, and

$$r_j(T) = \nabla_j(ab) - y^r \frac{\partial}{\partial y^j} \nabla_r(ab),$$

and taking into account that

$$\nabla_j a = 0,$$

we get

$$r_j(T) = ar_j(b) - a_{.j} \nabla_0 b,$$

where $\nabla_0 b = y^r \nabla_r b$.

The cubic space (M, \bar{F}) is a Berwald one if and only if the functions $2\bar{G}^i$ are polynomial in y^i . This is equivalent to the fact that the difference

$$2B^i := 2\bar{G}^i - 2G^i = -T^{ij} r_j(T)$$

is a polynomial function of degree 2 in y . There holds

Theorem 9 *The space $(M, F = \sqrt[3]{T})$, where T decomposes as a product*

$$T = a \cdot b \tag{12}$$

where $a = \gamma_{ij}(x) y^i y^j$ is a Riemannian metric and $b = b_i(x)$ is a 1-form, such that:

$$\|b\|^2 = \gamma^{ij} b_i b_j = 1$$

1. is of Berwald type, if and only if b is parallel with respect to a :

$$\nabla_i b_j = 0, \quad \forall i, j = 1, \dots, n.$$

Proof:

Let us suppose that $(M, \bar{F} = \sqrt[3]{ab})$ is Berwaldian and let us fix some arbitrary $x \in M$. Then $2B^i$ are polynomials of degree 2 and hence, so are $2B^i b_i$. By (11), we have $T^{ij} b_j = \frac{1}{2\Delta} (2bb^i - y^i)$, consequently,

$$\begin{aligned}
-2B^i b_i &= \frac{1}{2\Delta} (2bb^j - by^j) r_j(T) = \frac{1}{2\Delta} (2bb^j - y^j) (ar_j(b) - a_{.j} \nabla_0 b) = \\
&= \frac{1}{\Delta} (abb^j r_j(b) - 2b^2 \nabla_0 b + a \nabla_0 b).
\end{aligned}$$

But, $a - 2b^2 = 2b^2 - \Delta$, so we can write

$$-2B^i b_i = \frac{1}{\Delta} \{abb^j r_j(b) + (2b^2 - \Delta) \nabla_0 b\} = -\nabla_0 b + \frac{1}{\Delta} \{abb^j r_j(b) + 2b^2 \nabla_0 b\}.$$

Since the latter is a polynomial, Δ divides the polynomial $abb^j r_j(b) + 2b^2 \nabla_0 b = b(ab^j r_j(b) + 2b \nabla_0 b)$. Since a does not decompose in factors, a and b have no common factors; we notice that, in this case, b and Δ are also relatively prime, hence

$$\Delta \mid ab^j r_j(b) + 2b \nabla_0 b.$$

Again, we have $a = 4b^2 - \Delta$, and we get that $\Delta \mid 4b^2 b^j r_j(b) + 2b \nabla_0 b = 2b(2bb^j r_j(b) + \nabla_0 b)$, that is,

$$\Delta \mid (2bb^j r_j(b) + \nabla_0 b).$$

Both hand sides of the above are polynomials of degree 2 in y^i , hence there exists some $f = f(x)$ such that

$$(2bb^j r_j(b) + \nabla_0 b) = f(x) \Delta. \quad (13)$$

By identifying the coefficients in the above relation and taking into account that, by (11) $b^i \nabla_j b_i = 0$, we get

$$2b_i b^j \nabla_j b_r + 2b_r b^j \nabla_j b_i + \nabla_r b_i + \nabla_i b_r = f(x)(8b_i b_r - 2\gamma_{ir}).$$

Contracting with b^i and taking into account that $b^i b_i = 1$, the above leads to

$$b^i \nabla_i b_r = 2b_r f(x), \quad (14)$$

which yields

$$b^j r_j(b) = b^j (\nabla_j b_r - \nabla_r b_j) y^r = b^j \nabla_j b_0 = 2b f(x). \quad (15)$$

Replacing into (13), we have $4b^2 f(x) + \nabla_0 b = f(x) \Delta = f(x)(4b^2 - a)$; we obtain that

$$\nabla_0 b = -a f(x). \quad (16)$$

Let us come back now to the expression of $2B^i$:

$$-2B^i = T^{ij} (ar_j(b) - a_{.j} \nabla_0 b)$$

The last term, $T^{ij} a_{.j} \nabla_0 b$ is

$$T^{ij} a_{.j} \nabla_0 b = \frac{1}{\Delta} (2by^i - b^i a) \nabla_0 b = \frac{-a}{\Delta} (2by^i - b^i a) f(x).$$

The first one, $T^{ij} ar_j(b)$, is

$$\begin{aligned} T^{ij} ar_j(b) &= \frac{a}{2b\Delta} (\Delta \gamma^{ij} - 2bb^i y^j - 2bb^j y^i + ab^i b^j + y^i y^j) r_j(b) = \\ &= \frac{a}{2b\Delta} (\Delta \gamma^{ij} r_j(b) - 0 - 4b^2 y^i f(x) + 2abb^i f(x) + 0). \end{aligned}$$

Then,

$$-2B^i = \frac{a}{2b\Delta} \{ \Delta \gamma^{ij} r_j(b) - 4b^2 y^i f(x) + 2abb^i f(x) \} + \frac{2ab}{2b\Delta} (2by^i - b^i a) f(x)$$

The common denominator $2b\Delta$ has to divide the numerator. In particular, b has to divide the numerator. The only term which does not contain b explicitly as a factor is

$$a\Delta \gamma^{ij} r_j(b).$$

Since b has no common factors neither with a , nor with Δ , b has to divide the polynomial $\gamma^{ij} r_j(b)$ (of degree 1). That is, there exists some $\phi = \phi(x)$ such that $\gamma^{ij} r_j(b) = \phi^i(x)b$. Lowering the indices,

$$r_j(b) = \phi_j(x)b.$$

But, since $y^j r_j(b) = 0$, we get $0 = y^j r_j(b) = (y^j \phi_j(x))b$. Together with $b \neq 0$, this yields $y^j \phi_j(x) = 0$, or

$$\phi_j = 0,$$

which is nothing but $r_j(b) = 0$. The latter means actually

$$\nabla_r b_i - \nabla_i b_r = 0. \quad (17)$$

Let's now look at relation (14):

$$b^i \nabla_i b_r = 3b_r f(x), \quad (18)$$

By (17), it is equivalent to

$$b^i \nabla_r b_i = 3b_r f(x).$$

According to (11), we have $b^i \nabla_r b_i = 0$; the left hand side of the above is 0, hence

$$f(x) = 0,$$

which yields, together with (16),

$$\nabla_0 b = \nabla_j b_i y^i y^j = 0.$$

The latter, together with (17), leads to

$$\nabla_r b_i = 0,$$

q.e.d.

The converse statement is obvious.

Remark 10 *If (M, \bar{F}) is of Berwald type, then*

$$2B^i := 2\bar{G}^i - 2G^i = -T^{ij} r_j(T) = 0,$$

consequently, it has the same geodesics as the Riemannian space $(M, a = \gamma_{ij}(x)y^i y^j)$.

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